Interacting Field Theories in Robertson-Walker Spacetimes: Analytic Approximations

Carmen Molina-París,[1,2,5] **Paul R. Anderson,**[1,3]
and Stephen A. Ramsey[4]

The renormalization of a scalar field theory with a quartic self-coupling via adiabatic regularization in a Robertson-Walker spacetime is discussed. The adiabatic counterterms are presented in a way that is most conducive to numerical computations. A variation of the adiabatic regularization method is presented which leads to analytic approximations for the energy–momentum tensor of the quantum field and the quantum contribution to the effective mass of the mean field. Conservation of the energy–momentum tensor for the field is discussed and it is shown that the part of the energy–momentum tensor which depends only on the mean field is not conserved but the full renormalized energy–momentum tensor is conserved, as expected and required by the semiclassical Einstein's equation. It is also shown that if the analytic approximations are used the resulting approximate energy–momentum tensor is conserved. This allows a self-consistent backreaction calculation to be performed using the analytic approximations. The usefulness of the approximations is discussed.

1. INTRODUCTION

The study of free quantized fields in curved space has been a remarkably fruitful endeavor, particularly in the applications that have been made to black hole and cosmological spacetimes (see for example, Birrell and Davies, 1982). Much less has been done regarding interacting fields in curved space. However, interacting fields are very important since all real fields in nature appear to have interactions. Interactions also play an important role in cosmological models such as inflaton, being required in some cases for the inflation potential to have the right form and also for the thermalization that is necessary to reheat the universe

[1] Theoretical Division T-8, Los Alamos National Laboratory, Los Alamos, New Mexico.

[2] Centro de Astrobiología, CSIC/INTA, Carretera de Ajalvir Km. 4, Torrejón de Ardoz, Madrid, Spain.

[3] Department of Physics, Wake Forest University, Winston-Salem, North Carolina.

[4] Genome Center, University of Washington, Seattle, Washington.

[5] To whom correspondence should be addressed at Theoretical Division T-8, Los Alamos National Laboratory, Los Alamos, New Mexico, 87545; e-mail: carmen@t6-serv.lanl.gov.

after inflation. Interactions can also significantly enhance the particle production that often occurs for free fields in curved space.

The study of interacting quantum fields in Robertson-Walker (RW) space-times is of great importance as well in understanding quantum fields in Minkowski space. It is well known that following a relativistic heavy ion collision the quark-gluon plasma produced eventually undergoes a chiral phase transition. A good approximation to describing the dynamics of this system is provided by the linear σ-model and by assuming that the expansion is mostly radial (Cooper *et al.*, 1995; Lampert *et al.*, 1996). Written in terms of the spherical hydrodynamical fluid coordinates the system is equivalent to an interacting quantum scalar field (mean field plus fluctuations) in a RW spacetime (Lampert and Molina-París, 1998).

Perhaps the simplest interacting quantum field theory in four dimensions is a scalar field with a quartic self-coupling, often called the "$\lambda \phi^4$" theory. There is a long history of the study of this theory in curved space. The original investigations centered on renormalization. Drummond (1975), Birrell and Ford (1979, 1980), Bunch *et al.* (1980), and Bunch and Panangaden (1980) investigated the renormalization of the theory in various cosmological spacetimes using techniques such as dimensional regularization. Bunch and Parker (1980) showed that the theory is renormalizable in an arbitrary spacetime to second order in the coupling constant λ. Birrell (1980) extended their work by using momentum space techniques and computing self-energy graphs to second order in λ.

Along with studies of the renormalization of the theory, various calculations have been undertaken. For example, Ford and Toms investigated phase transitions caused by one-loop radiative corrections in an expanding universe (Ford and Toms, 1982). The one-loop finite temperature effective potential for a $\lambda \phi^4$ theory in a RW universe was calculated by Hu, under the assumptions that the rate of change of quantum fluctuations is much greater than that of the mean field and the expansion rate of the universe (Hu, 1983). Ringwald investigated the evolution of the expectation value of the quantum fluctuation $\langle \psi^2 \rangle$ at one-loop order in a spatially flat RW universe (Ringwald, 1987).

The quantity $\langle \psi^2 \rangle$ can be used to determine the backreaction of the quantum fluctuations on the mean (or classical) scalar field as it appears as an effective mass for the mean field at one-loop order. However, to determine the backreaction of the scalar field on the spacetime geometry, one must compute the renormalized energy–momentum tensor for the field. The renormalization of the energy–momentum tensor to one-loop order in a spatially flat RW spacetime was discussed by Paz and Mazzitelli (1988). They displayed the renormalization counterterms which were obtained using point splitting, adiabatic regularization, and dimensional regularization. The divergent counterterms were displayed in the format of dimensional regularization. Mazzitelli *et al.* used this formalism in a calculation relating to the evolution of the inflaton and the reheating after inflation in the new inflationary scenario (Mazzitelli *et al.*, 1989).

There are two ways that so called "nonperturbative" effects are usually taken into account. One is the Hartree approximation which works for a single scalar field (Jackiw and Kerman, 1979; Stevenson, 1985). The other is the large N approximation where a single scalar field is replaced with N scalar fields which are coupled via a quartic interaction, which is invariant under the group $O(N)$ and is thus often called the $O(N)$ model. Mazzitelli and Paz considered the renormalization in both the Hartree and large N approximations in an arbitrary background gravitational field, using point splitting techniques and adiabatic and dimensional regularization (Mazzitelli and Paz, 1989). More recent work on the backreaction of a scalar field on the background geometry in a RW spacetime has been done for inflationary models by Boyanovsky *et al.* (n.d., 1995, 1997) and by Ramsey and Hu (1997a,b).

In this paper we derive a set of renormalized equations that can be used to determine the evolution of a quartically coupled scalar field with arbitrary mass and curvature coupling to one-loop order in a RW spacetime. We also derive expressions for the unique components of the renormalized energy–momentum tensor that can be used to determine the backreaction of the field on the spacetime geometry. To renormalize we use the method of adiabatic regularization (Fulling and Parker, 1974; Fulling *et al.*, 1974; Parker, 1966; Parker and Fulling, 1974) which is particularly useful for deriving a set of equations which are to be solved numerically. We display the counterterms for the energy–momentum tensor and the quantum contribution to the effective mass of the mean field (for arbitrary mass and curvature coupling). Our results are summarized in Molina-París *et al.* (2000). Previously the adiabatic counterterms for the energy–momentum tensor have been displayed by Paz and Mazzitelli (1988) but only in the context of dimensional regularization, which makes it difficult to use them for numerical computations, and by Ramsey and Hu for the minimally coupled case (if one takes the one-loop limit of their $1/N$ expansion) (Ramsey and Hu, 1997a,b).

We discuss in more detail than has previously been done the conservation of the energy–momentum tensor for the full system (mean field plus quantum fluctuations). We show that the natural division of this tensor into a "classical" and a "quantum" piece leads to neither piece being separately conserved. We also show explicitly that the full energy–momentum tensor is conserved.

We present a variation on the method of adiabatic regularization which has been used by Anderson and Eaker to develop an analytic approximation for the energy–momentum tensor for a free scalar field in a RW spacetime (Anderson and Eaker, 2000). We use this method to derive analytic approximations to both the energy–momentum tensor of the quantum fluctuation and to the effective mass of the mean field. If the analytic approximation is used in the equation for the mean field and if it is also used for the "quantum" energy–momentum tensor, then the resulting set of equations results in a conserved approximate energy–momentum tensor. Thus the analytic approximations can be used in lieu of the

full renormalized expressions in the mean field and semiclassical backreaction equations. They are useful for the investigation of vacuum polarization effects, but not particle production since particle production is a nonlocal phenomenon. Nevertheless, the approximations give important information, particularly if one wishes to estimate the conditions under which the loop expansion breaks down. We discuss the validity of the approximation and argue that it is likely to be most useful for massless fields.

It is well known that the problem of solving Einstein's equation in the presence of quantum matter is not an easy one. On the left-hand-side of Einstein's equation one needs higher order geometric tensors ($^{(1)}H_{\mu\nu}$, $^{(2)}H_{\mu\nu}$, and $H_{\mu\nu}$) that involve up to fourth order time derivatives of the metric $g_{\mu\nu}$ (Birrell and Davies, 1982). On the right-hand-side one encounters the expectation value of the energy–momentum tensor of the quantum field in a certain quantum state (for which there is no a priori rule to be determined or chosen), a quantity that is ultraviolet divergent. One has to regularize and renormalize $\langle T_{\mu\nu} \rangle$ in such a way that it remains covariantly conserved, as the left-hand-side of Einstein's equation is. This is why the problem of a full backreaction of the quantum matter on the spacetime geometry is so difficult. One has to make use of regularization methods that are suited for a numeric computation (as the dynamical equation for the quantum field is most likely not to have analytic solutions), and at the same time guarantee the covariant conservation of the renormalized value of $\langle T_{\mu\nu} \rangle$. In this paper we use adiabatic subtraction to fulfill both requirements and to set up the formalism and techniques required to perform a backreaction calculation for an interacting theory (mean field plus fluctuations) in a general RW spacetime.

Before attempting to solve the semiclassical backreaction equations, it will be helpful to estimate how big or small the quantum effects on the geometry and the mean field are. As a means of doing so we derive analytic approximations for the energy–momentum tensor and the quantum contribution to the effective mass of the field in an arbitrary RW spacetime. This is a generalization of the approximation found by Anderson and Eaker (2000) for the energy–momentum tensor of the free scalar field. It yields a renormalized and covariantly conserved approximate energy–momentum tensor for the quantum fluctuations (that carries no information, whatsoever, about particle production effects). This paper presents all the technical details for such a calculation. Future work will consist of evaluating both the exact and the analytically approximated energy–momentum tensor of the quantum fluctuations in various scenarios, such as the reheating period of the inflationary regime of the early universe and the spherical expansion of the quark-gluon plasma.

In Section 2 we derive the one-loop equations for the mean and quantum fields. We also compute the energy–momentum tensor of the system at one-loop and show that it splits naturally into two terms: a "classical" and a "quantum" energy–momentum tensor. In Section 3 we discuss the method of adiabatic regularization

when the quantum fluctuations have a time dependent mass and derive the adiabatic order two and four counterterms that need to be subtracted from $\langle \psi^2 \rangle_B$ and $\langle T_{\mu\nu} \rangle_B$, respectively. We explicitly separate those new terms that were not present in the free case (Anderson and Parker, 1987; Bunch, 1980). In Section 4 we introduce the analytic approximation as a way of estimating the importance of vacuum polarization effects and as a first approximation to doing a full back-reaction calculation. In Section 5 we discuss the covariant conservation of the energy–momentum tensor. We show that the part of the energy–momentum tensor that depends only on the mean field is usually not conserved by itself, but that both the full energy–momentum tensor and its analytic approximation are covariantly conserved. A brief summary of our results is given in Section 6. In the Appendices A and B we show explicitly the conservation of the bare and the renormalized energy–momentum tensor, respectively.

2. BACKGROUND, CONVENTIONS, AND NOTATION

We consider a quantum scalar field with quartic self-interactions in a RW spacetime. The metric of a RW spacetime can be written in the form[6]

$$ds^2 = a^2(\eta) \left[d\eta^2 - \frac{dr^2}{1 - \kappa r^2} - r^2 \, d\Omega^2 \right], \tag{2.1}$$

where η is the conformal time coordinate and $\kappa = -1, 0, +1$ is the three-dimensional spatial curvature, corresponding to spatial Cauchy hypersurfaces that have negative, zero, and positive spatial curvature, respectively.

The action of a scalar field with a quartic self-interaction is given by

$$S_{\text{matter}}[\Phi, g_{\mu\nu}] = -\frac{1}{2} \int d^4x (-g)^{1/2} \left[\Phi(\Box + m^2 + \xi R)\Phi + \frac{\lambda}{12} \Phi^4 \right], \tag{2.2}$$

where g is the determinant of the metric, $\Box$ the D'Alembert wave operator given by $\Box = g^{\mu\nu} \nabla_\mu \nabla_\nu$, and R the scalar curvature of the RW spacetime.

The equation of motion for the classical field (obtained by the principle of least action) is given by

$$\left(\Box + m^2 + \xi R + \frac{\lambda}{3!} \Phi^2 \right) \Phi = 0. \tag{2.3}$$

The classical energy–momentum tensor is

$$T_{\mu\nu} = (1 - 2\xi)\partial_\mu \Phi \partial_\nu \Phi + (2\xi - 1/2)g_{\mu\nu}\partial_\alpha \Phi \partial^\alpha \Phi - 2\xi \Phi \nabla_\mu \nabla_\nu \Phi$$

$$+ 2\xi g_{\mu\nu} \Phi \Box \Phi - \xi G_{\mu\nu} \Phi^2 + \frac{m^2}{2} g_{\mu\nu} \Phi^2 + \frac{\lambda}{4!} g_{\mu\nu} \Phi^4. \tag{2.4}$$

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[6] Throughout this paper we use units such that $\hbar = c = 1$. The metric signature is $(+ - - -)$ and the conventions for the curvature tensors are $R^\alpha_{\beta\gamma\delta} = \Gamma^\alpha_{\beta\gamma,\delta} - \ldots$ and $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu}$.

If we quantize the theory the classical field Φ becomes an operator. We then define the mean (or background) field ϕ by the equations

$$\Phi \equiv \phi + \psi, \tag{2.5a}$$

$$\phi \equiv \langle\Phi\rangle, \tag{2.5b}$$

where the expectation value is taken with respect to the initial quantum state of the system (in the Heisenberg representation).

Taking the expectation value of Eq. (2.3) and noting that

$$\langle\Phi^3\rangle = \phi^3 + 3\phi^2\langle\psi\rangle + 3\phi\langle\psi^2\rangle + \langle\psi^3\rangle = \phi^3 + 3\phi\langle\psi^2\rangle + \langle\psi^3\rangle, \tag{2.6}$$

we find the following equation for the mean field ϕ

$$(\Box + m^2 + \xi R)\phi + \frac{\lambda}{3!}(\phi^3 + 3\phi\langle\psi^2\rangle + \langle\psi^3\rangle) = 0. \tag{2.7}$$

In the same way, by subtracting Eq. (2.7) from Eq. (2.3), we obtain the equation of motion for the quantum fluctuation ψ

$$(\Box + m^2 + \xi R)\psi + \frac{\lambda}{3!}(3\phi^2\psi - 3\phi\langle\psi^2\rangle + 3\phi\psi^2 + \psi^3 - \langle\psi^3\rangle) = 0. \tag{2.8}$$

If we truncate at one-loop (free field theory for the quantum fluctuation ψ), the equations of motion (2.7) and (2.8) become (Paz and Mazzitelli, 1988)

$$(\Box + m^2 + \xi R)\phi + \frac{\lambda}{3!}\phi^3 + \frac{\lambda}{2}\phi\langle\psi^2\rangle = 0, \tag{2.9a}$$

$$(\Box + m^2 + \xi R)\psi + \frac{\lambda}{2}\phi^2\psi = 0. \tag{2.9b}$$

The expectation value of the energy–momentum tensor can be broken into a "classical" and a "quantum" part. The classical part is given by

$$\langle T_{\mu\nu}\rangle^C \equiv (1 - 2\xi)\partial_\mu\phi\partial_\nu\phi + (2\xi - 1/2)g_{\mu\nu}\partial_\alpha\phi\partial^\alpha\phi - 2\xi\phi\nabla_\mu\nabla_\nu\phi$$

$$+ 2\xi g_{\mu\nu}\phi\Box\phi - \xi G_{\mu\nu}\phi^2 + \frac{m^2}{2}g_{\mu\nu}\phi^2 + \frac{\lambda}{4!}g_{\mu\nu}\phi^4, \tag{2.10a}$$

while the quantum part is

$$\langle T_{\mu\nu}\rangle^Q = \langle T_{\mu\nu}\rangle_B \equiv (1 - 2\xi)\langle\partial_\mu\psi\partial_\nu\psi\rangle + (2\xi - 1/2)g_{\mu\nu}\langle\partial_\alpha\psi\partial^\alpha\psi\rangle$$

$$- 2\xi\langle\psi\nabla_\mu\nabla_\nu\psi\rangle + 2\xi g_{\mu\nu}\langle\psi\Box\psi\rangle - \xi G_{\mu\nu}\langle\psi^2\rangle$$

$$+ \frac{m^2}{2}g_{\mu\nu}\langle\psi^2\rangle + \frac{\lambda}{4}g_{\mu\nu}\phi^2\langle\psi^2\rangle. \tag{2.10b}$$

Equations (2.9a)–(2.10b) describe our system at one-loop order [mean field ϕ plus quantum fluctuations ψ, which contribute to the effective mass of ϕ, see Eq. (2.7)].

It is well known (Bunch, 1980; Collins, 1974) that to make sense of these equations one needs to regularize the theory, that is, on the one hand, define a way to obtain the renormalized parameters (m_R, λ_R, and ξ_R) from the bare ones (m, λ, and ξ), and on the other hand, regularize the divergent quantities $\langle \psi^2 \rangle_B$ and $\langle T_{\mu\nu} \rangle_B$, to obtain the physically finite energy–momentum tensor of the system. In the next section we discuss these issues.

3. ADIABATIC REGULARIZATION

It has been shown (using dimensional regularization) that for a $\lambda\phi^4$ theory in a general spacetime, the bare and the renormalized parameters are related in the following way (Bunch and Panangaden, 1980; Bunch *et al.*, 1980)

$$m_B^2 \equiv m_R^2 - \frac{3\lambda_R}{8\pi^2(n-4)}m_R^2,$$

$$\xi_B - \frac{1}{6} \equiv \xi_R - \frac{1}{6} - \frac{3\lambda_R}{8\pi^2(n-4)}\left(\xi_R - \frac{1}{6}\right),$$

$$\lambda_B \equiv \lambda_R - \frac{9\lambda_R^3}{8\pi^2(n-4)}. \tag{3.1}$$

Here n is the number of dimensions the spacetime has been analytically continued to. Thus, even for a general spacetime the counterterms for m^2, ξ, and λ are constant in space and time as expected.

In principle one would prefer to renormalize at the level of the effective action and then vary that action with respect to ϕ and the metric $g_{\mu\nu}$ to obtain the renormalized equations of motion and energy–momentum tensor, respectively (Cognola, 1994). However, the computation of the one-loop effective action for arbitrary mean fields ϕ and arbitrary spacetime geometries is quite involved and will not yield an intrinsically different answer from that obtained by looking at the one-loop field equations. For this reason, we consider a different method called adiabatic regularization which works at the level of the field equations (Fulling and Parker, 1974; Fulling *et al.*, 1974; Parker, 1966; Parker and Fulling, 1974). Another advantage of adiabatic regularization is that it is well suited to perform numerical calculations (Anderson, 1985, 1986; Suen and Anderson, 1987). In adiabatic regularization the divergences in quantities such as $\langle \psi^2 \rangle_B$ and $\langle T_{\mu\nu} \rangle_B$ are computed using a WKB expansion for the modes of the quantized field ψ. These terms are then subtracted from the unrenormalized (bare) expressions with the result that

$$\langle \psi^2 \rangle_R = \langle \psi^2 \rangle_B - \langle \psi^2 \rangle_{ad}, \tag{3.2a}$$

$$\langle T_{\mu\nu} \rangle_R = \langle T_{\mu\nu} \rangle_B - \langle T_{\mu\nu} \rangle_{ad}, \tag{3.2b}$$

where the subscripts R, B, and ad stand for the renormalized, bare, and the adiabatic values, respectively, of $\langle \psi^2 \rangle$ and $\langle T_{\mu\nu} \rangle$. This procedure has been shown to be

equivalent to point splitting for free scalar fields in a RW spacetime (Anderson and Parker, 1987; Birrell, 1978). For the quartically coupled scalar field Eqs. (2.9a) and (2.9b) then become

$$\left(\Box + m_R^2 + \xi_R R\right)\phi + \frac{\lambda_R}{3!}\phi^3 + \frac{\lambda_R}{2}\phi\langle\psi^2\rangle_R = 0, \tag{3.2c}$$

$$\left(\Box + m_R^2 + \xi_R R\right)\psi + \frac{\lambda_R}{2}\phi^2\psi = 0. \tag{3.2d}$$

In what follows we only consider the renormalized values of the coupling constants m, ξ, and λ, so we drop the subscript R from these quantities.

We assume that the mean field is homogeneous $\phi = \phi(\eta)$, as a RW spacetime is homogeneous and isotropic. Then Eqs. (3.2c) and (3.2d) become

$$\phi'' + 2\frac{a'}{a}\phi' + a^2\left(m^2 + \xi R + \frac{\lambda}{3!}\phi^2 + \frac{\lambda}{2}\langle\psi^2\rangle_R\right)\phi = 0. \tag{3.3a}$$

$$\psi'' + 2\frac{a'}{a}\psi' - \Delta^{(3)}\psi + a^2\left(m^2 + \xi R + \frac{\lambda}{2}\phi\right)\psi = 0. \tag{3.3b}$$

Here primes denote derivatives with respect to η. The $\eta\eta$ component of the classical renormalized energy–momentum tensor (2.10a) is given by

$$\langle T^\eta{}_\eta\rangle_R^C = \frac{1}{2a^2}\phi'\phi' + \frac{6}{a^2}\xi\frac{a'}{a}\phi\phi' + \frac{3}{a^2}\xi\left(\frac{a'^2}{a^2} + \kappa\right)\phi^2 + \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4, \tag{3.3c}$$

and the trace is

$$\langle T\rangle_R^C = (6\xi - 1)\frac{1}{a^2}\phi'\phi' + \frac{6}{a^2}\xi\phi\phi'' + \frac{12}{a^2}\xi\frac{a'}{a}\phi\phi' + \xi R\phi^2 + 2m^2\phi^2 + \frac{\lambda}{3!}\phi^4. \tag{3.3d}$$

In order to determine the renormalization counterterms used in adiabatic regularization we first review canonical quantization in a RW spacetime. We then discuss the WBK expansion for the modes of the quantum fluctuation field ψ and compute the adiabatic counterterms needed to renormalize $\langle\psi^2\rangle_B$ and $\langle T_{\mu\nu}\rangle_B$.

Since at one-loop order the quantum field ψ is a free field with an effective mass of the form $m^2 + \frac{\lambda\phi^2}{2}$, [see Eq. (3.3b)], it can be expanded in the following manner (Birrell and Davies, 1982)

$$\psi(x) = \frac{1}{a(\eta)}\int d\tilde\mu(\mathbf{k})[a_\mathbf{k}Y_\mathbf{k}(\mathbf{x})f_k(\eta) + a_\mathbf{k}^\dagger Y_\mathbf{k}^*(\mathbf{x})f_k^*(\eta)], \tag{3.4}$$

where the measure is given by (Bunch, 1980)

$$\int d\tilde{\mu}(\mathbf{k}) \equiv \int d^3\mathbf{k} \qquad \text{if } \kappa = 0,$$

$$\equiv \int_0^{+\infty} dk \sum_{l,m} \quad \text{if } \kappa = -1,$$

$$\equiv \sum_{k,l,m} \qquad \text{if } \kappa = +1.$$

The spatial part of the mode function, $Y_{\mathbf{k}}(\mathbf{x})$, obeys the equation

$$\Delta^{(3)} Y_{\mathbf{k}}(\mathbf{x}) = -(k^2 - \kappa) Y_{\mathbf{k}}(\mathbf{x}), \tag{3.5}$$

and the time dependent part $f_k(\eta)$ is a solution to the equation

$$f_k'' + \left[\omega_k^2 + \frac{\lambda}{2} a^2 \phi^2 + (\xi - 1/6) a^2 R \right] f_k = 0. \tag{3.6}$$

Here $\omega_k^2 \equiv k^2 + m^2 a^2$, primes denote derivatives with respect to the conformal time η, and for spacetimes with the metric (2.1) the scalar curvature is given by

$$R = 6 \left(\frac{a''}{a^3} + \frac{\kappa}{a^2} \right). \tag{3.7}$$

The bare expression for the quantum part of the energy–momentum tensor (2.10b) is (Anderson and Parker, 1987; Bunch, 1980)

$$\langle T^\eta{}_\eta \rangle_B^Q = \frac{1}{4\pi^2 a^4} \int d\mu(k) \left\{ |f_k'|^2 + \left(k^2 + m^2 a^2 + \frac{\lambda}{2} a^2 \phi^2 \right) |f_k|^2 \right.$$
$$\left. + 6 \left(\xi - \frac{1}{6} \right) \left[\frac{a'}{a} (f_k f_k^{*\prime} + f_k^* f_k') - \left(\frac{a'^2}{a^2} - \kappa \right) |f_k|^2 \right] \right\}, \tag{3.8a}$$

$$\langle T \rangle_B^Q = \frac{1}{2\pi^2 a^4} \int d\mu(k) \left\{ \left(m^2 a^2 + \frac{\lambda}{2} a^2 \phi^2 \right) |f_k|^2 + 6 \left(\xi - \frac{1}{6} \right) \right.$$
$$\times \left[|f_k'|^2 - \frac{a'}{a} (f_k f_k^{*\prime} + f_k^* f_k') - \left(k^2 + m^2 a^2 + \frac{\lambda}{2} a^2 \phi^2 \right. \right.$$
$$\left. \left. \left. + \frac{a''}{a} - \frac{a'^2}{a^2} + \left(\xi - \frac{1}{6} \right) a^2 R \right) |f_k|^2 \right] \right\}. \tag{3.8b}$$

Note that the pressure is not an independent quantity and can be obtained from knowledge of the energy density and the trace.

The equation for the mean field (2.9a) also contains the quantity $\langle \psi^2 \rangle$. Its bare expression can be written in terms of the mode functions as follows

$$\langle \psi^2 \rangle_B = \frac{1}{2\pi^2 a^2} \int d\mu(k) |f_k|^2. \tag{3.8c}$$

In these expressions the measure is given by

$$\int d\mu(k) \equiv \int_0^{+\infty} dk \, k^2 \quad \text{if } \kappa = 0, -1,$$

$$\equiv \sum_{k=1}^{+\infty} k^2 \qquad \text{if } \kappa = +1.$$

To determine the adiabatic counterterms needed to renormalize these expectation values we solve the mode equation (3.6) using a WKB expansion. To obtain this expansion we first make the variable transformation

$$f_k = (2W_k)^{-1/2} \exp\left[\int^\eta d\eta' \, W_k(\eta') \right]. \tag{3.9}$$

Substituting Eq. (3.9) into Eq. (3.6) yields

$$W_k^2 = \omega_k^2 + \frac{\lambda}{2} a^2 \phi^2 + \left(\xi - \frac{1}{6} \right) a^2 R - \frac{1}{2} \left(\frac{W_k''}{W_k} - \frac{3}{2} \frac{W_k'^2}{W_k^2} \right). \tag{3.10}$$

This equation is then solved iteratively with ω_k being of adiabatic order zero and the next two terms on the right-hand-side being of adiabatic order two.[7] Thus to second adiabatic order the solution for W_k is

$$W_k = \omega_k + \frac{1}{2\omega_k} \left[\frac{\lambda}{2} a^2 \phi^2 + \left(\xi - \frac{1}{6} \right) a^2 R - \frac{1}{2} \left(\frac{\omega_k''}{\omega_k} - \frac{3}{2} \frac{\omega_k'^2}{\omega_k^2} \right) \right]. \tag{3.11}$$

To renormalize $\langle \psi^2 \rangle_B$ we use a second order adiabatic expansion for the modes. A fourth-order expansion is necessary to cancel the divergences in $\langle T_{\mu\nu} \rangle_B$ (Anderson and Parker, 1987; Bunch, 1980). The renormalization counterterms for these quantities are

$$\langle \psi^2 \rangle_{ad} = \langle \psi^2 \rangle_{ad}^F - \frac{1}{4\pi^2 a^2} \int d\mu(k) \frac{\lambda \phi^2 a^2}{4\omega_k^3}, \tag{3.12a}$$

$$\langle T^\eta{}_\eta \rangle_{ad} = \langle T^\eta{}_\eta \rangle_{ad}^F + \frac{1}{4\pi^2 a^4} \int d\mu(k) \left\{ \frac{\lambda \phi^2 a^2}{4\omega_k} - \frac{\lambda^2 \phi^4 a^4}{32\omega_k^3} \right.$$

$$+ \frac{m^2 \lambda (\phi^2 a^2 a'^2 + \phi\phi' a^3 a')}{8\omega_k^5} - \frac{5m^4 \lambda \phi^2 a^4 a'^2}{32\omega_k^7} + 6 \left(\xi - \frac{1}{6} \right)$$

$$\times \left[-\frac{\lambda (\phi^2 a'^2 + 2\phi\phi' aa' + \kappa\phi^2 a^2)}{8\omega_k^3} + \frac{3m^2 \lambda \phi^2 a^2 a'^2}{8\omega_k^5} \right] \right\}, \tag{3.12b}$$

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[7] The $\frac{\lambda}{2} a^2 \phi^2$ term is considered to be of second adiabatic order because only terms with up to two time derivatives of ϕ are needed to cancel divergences in $\langle T_{\mu\nu} \rangle_B$ (Paz and Mazzitelli, 1988).

$$\langle T \rangle_{ad} = \langle T \rangle_{ad}^F + \frac{1}{4\pi^2 a^4} \int d\mu(k) \left\{ \frac{\lambda\phi^2 a^2}{2\omega_k} - \frac{\lambda\phi^2 a^4(2m^2 + \lambda\phi^2)}{8\omega_k^3} \right.$$

$$+ \frac{m^2\lambda}{32\omega_k^5} (3\lambda\phi^4 a^6 + 4a^4\phi\phi'' + 4a^4\phi'^2 + 8\phi^2 a^2 a'^2 + 8\phi^2 a^3 a''$$

$$+ 16\phi\phi' a^3 a') - \frac{5m^4\lambda}{16\omega_k^7} (4\phi^2 a^4 a'^2 + \phi^2 a^5 a'' + 2\phi\phi' a^5 a')$$

$$+ \frac{35m^6\lambda a^6 a'^2}{32\omega_k^9} + 6\left(\xi - \frac{1}{6}\right) \left[-\frac{\lambda}{4\omega_k^3} (2\phi\phi' aa' + \phi^2 aa'' + \phi\phi'' a^2 \right.$$

$$+ \phi'^2 a^2 + \kappa\phi^2 a^2) + \frac{3m^2\lambda}{8\omega_k^5} (3\phi^2 a^2 a'^2 + 4\phi\phi' a^3 a' + 2\phi^2 a^3 a''$$

$$\left. \left. + \kappa\phi^2 a^4) - \frac{15m^4\lambda\phi^2 a^4 a'^2}{8\omega_k^7} \right] \right\}. \tag{3.12c}$$

The adiabatic counterterms for the free field are (Anderson and Parker, 1987; Bunch, 1980)

$$\langle \psi^2 \rangle_{ad}^F = \frac{1}{4\pi^2 a^2} \int d\mu(k) \left[\frac{1}{\omega_k} - \left(\xi - \frac{1}{6}\right) \frac{a^2 R}{2\omega_k^3} + \frac{m^2}{4\omega_k^5} \right.$$

$$\left. \times (a'^2 + aa'') - \frac{5m^4 a^2 a'^2}{8\omega_k^7} \right], \tag{3.12d}$$

$$\langle T^\eta{}_\eta \rangle_{ad}^F = \frac{1}{4\pi^2 a^4} \int d\mu(k) \left\{ \omega_k + \frac{m^4 a^2 a'^2}{8\omega_k^5} - \frac{m^4}{32\omega_k^7} (2a^2 a' a''' - a^2 a''^2 \right.$$

$$+ 4aa'^2 a'' - a'^4) + \frac{7m^6 a^2}{16\omega_k^9} (aa'^2 a'' + a'^4) - \frac{105m^8 a^4 a'^4}{128\omega_k^{11}} + \left(\xi - \frac{1}{6}\right)$$

$$\times \left[-\frac{3}{\omega_k}\left(\frac{a'^2}{a^2} - \kappa\right) - \frac{3m^2 a'^2}{\omega_k^3} + \frac{3m^2}{4\omega_k^5}\left(2a' a''' - a''^2 - \frac{a'^4}{a^2}\right) \right.$$

$$\left. - \frac{15m^2}{8\omega_k^7} (4aa'^2 a'' + 3a'^4 + \kappa a^2 a'^2) + \frac{105m^6 a^2 a'^4}{8\omega_k^9} \right] + \left(\xi - \frac{1}{6}\right)^2$$

$$\times \left[-\frac{9}{2\omega_k^3}\left(\frac{2a' a'''}{a^2} - \frac{a''^2}{a^2} - \frac{4a'^2 a''}{a^3} - \frac{2\kappa a'^2}{a^2} + \kappa^2\right) \right.$$

$$\left. \left. + \frac{27m^2}{\omega_k^5}\left(\frac{a'^2 a''}{a} + \kappa a'^2\right) \right] \right\}, \tag{3.12e}$$

$$\langle T \rangle_{ad}^F = \frac{1}{4\pi^2 a^4} \int d\mu(k) \left\{ \frac{m^2 a^2}{\omega_k} + \frac{m^4 a^2}{4\omega_k^5}(aa'' + a'^2) - \frac{5m^6 a^4 a'^2}{8\omega_k^7} \right.$$

$$- \frac{m^4 a^2}{16\omega_k^7}(aa'''' + 4a'a''' + 3a''^2) + \frac{7m^6 a^2}{32\omega_k^9}(4a^2 a'a''' + 18aa'^2 a''$$

$$+ 3a^2 a''^2 + 3a'^4) - \frac{231m^8 a^4}{32\omega_k^{11}}(aa'^2 a'' + a'^4) + \frac{1155m^{10} a^6 a'^4}{128\omega_k^{13}}$$

$$+ \left(\xi - \frac{1}{6} \right) \left[-\frac{6}{\omega_k} \left(\frac{a''}{a} - \frac{a'^2}{a^2} \right) - \frac{3m^2}{\omega_k^3}(2aa'' - a'^2 + \kappa a^2) \right.$$

$$+ \frac{9m^4 a^2 a'^2}{\omega_k^5} + \frac{3m^2}{2\omega_k^5} \left(aa'''' - 2\frac{a'^2 a''}{a} + \frac{a'^4}{a^2} \right) - \frac{15m^4}{4\omega_k^7}(4a^2 a'a'''$$

$$+ 3a^2 a''^2 + 8aa'^2 a'' - a'^4 + \kappa a^3 a'' + \kappa a^2 a'^2) + \frac{105m^6 a^2}{8\omega_k^9}(8aa'^2 a''$$

$$+ 5a'^4 + \kappa a^2 a'^2) - \frac{945m^8 a^4 a'^4}{8\omega_k^{11}} \right] + \left(\xi - \frac{1}{6} \right)^2 \left[-\frac{9}{\omega_k^3} \left(\frac{a''''}{a} - \frac{4a'a'''}{a^2} \right. \right.$$

$$- \frac{3a''^2}{a^2} + \frac{6a'^2 a''}{a^3} - \frac{2\kappa a''}{a} + \frac{2\kappa a'^2}{a^2} \right) + \frac{27m^2}{2\omega_k^5} \left(4a'a''' + 3a''^2 \right.$$

$$\left. - 6\frac{a'^2 a''}{a} + 4\kappa aa'' - 2\kappa a'^2 + a^2 \kappa^2 \right) - \frac{135m^4}{\omega_k^7}(aa'^2 a'' + \kappa a^2 a'^2) \right] \right\}.$$

$$(3.12f)$$

Equations (3.12b)–(3.12c) [together with (3.12d)–(3.12f)] give the adiabatic counterterms needed to obtain the renormalized values of $\langle \psi^2 \rangle$ and $\langle T_{\mu\nu} \rangle$ for a $\lambda \phi^4$-interacting quantum scalar field in a RW spacetime, for the case of a homogeneous mean field $\phi(\eta)$.[8]

4. ANALYTIC APPROXIMATIONS

As it stands the method of adiabatic regularization can be used to compute the quantities $\langle \psi^2 \rangle_R$ and $\langle T_{\mu\nu} \rangle_R$. This then allows one to obtain a self-consistent solution to the mean field and mode equations (2.9a) and (2.9b) in a background RW spacetime (in the test field approximation) or to these equations plus the semiclassical backreaction equations in a RW spacetime. But before getting involved

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[8] Mazzitelli and Paz presented these counterterms with the points separated and with the divergence structure given in the context of dimensional regularization (Paz and Mazzitelli, 1988). Ramsey and Hu carried out the adiabatic expansion up to order four for $\xi_R = 0$ in the context of the leading $1/N$ expansion (Ramsey and Hu, 1997a,b).

in the backreaction problem (and the difficulties this presents), it would be very useful to have a way of estimating how big or small the quantum corrections are beyond test field approximation. In this section we discuss this issue and present such an approximation.

For a free quantum scalar field in a RW spacetime, Anderson and Eaker have shown that there is a way of defining a certain approximate energy–momentum tensor that is covariantly conserved (Anderson and Eaker, 2000). In the present paper we extend this method to interacting quantum fields and discuss its advantages and limitations.

The analytic approximations result from a revision of the method of adiabatic regularization that simplifies the calculations in a RW spacetime with compact spatial sections, and in the process yields approximations for the quantities $\langle \psi^2 \rangle_R$ and $\langle T_{\mu\nu} \rangle_R$. These analytic approximations give information about vacuum polarization effects but not particle production, since particle production is a nonlocal phenomenon. However, they also make it possible to solve the mean field, mode, and semiclassical backreaction equations in an approximate manner, which goes beyond the test field approximation and does not involve a full backreaction calculation. This can be useful, for example, if one wishes to determine under what conditions vacuum polarization effects will be important and what influence they may have on the mean field and the spacetime geometry. In particular, we believe that these analytic approximations may provide important information for reheating calculations, just before particle production from the inflaton field takes place. The analytic approximations provide a natural way to estimate the change in the vacuum polarization energy of the inflaton field. During the inflationary regime the mean field ϕ dominates the energy density of the universe. It is reasonable to expect that when the vacuum polarization energy is of the order of the mean field energy density, the inflaton field will switch from the slow-roll regime to the oscillatory behavior, that will eventually lead to particle production. In this paper we restrict ourselves to presenting the analytic approximations and our method for obtaining them.

In order to derive the analytic approximation we first improve on the method of adiabatic regularization by expanding the renormalization counterterms in inverse powers of k, keeping only terms which are ultraviolet divergent. For the case of compact spatial sections ($\kappa = +1$) the integral is also changed into a sum. We call the resulting expressions $\langle \psi^2 \rangle_d$ and $\langle T_{\mu\nu} \rangle_d$, respectively. In a general RW spacetime they have the form

$$
\langle \psi^2 \rangle_d = \frac{1}{4\pi^2 a^2} \int d\mu(k) \frac{1}{k} - \frac{1}{4\pi^2 a^2} \int d\bar{\mu}(k) \frac{1}{k^3}
$$
$$
\times \left[\frac{m^2 a^2}{2} + \frac{\lambda \phi^2 a^2}{4} + \left(\xi - \frac{1}{6} \right) \frac{a^2 R}{2} \right], \tag{4.1a}
$$

$$\langle T^\eta{}_\eta\rangle_d = \frac{1}{4\pi^2 a^4}\int d\mu(k)\left\{k+\frac{1}{k}\left[\frac{m^2 a^2}{2}+\frac{\lambda\phi^2 a^2}{4}-3\left(\xi-\frac{1}{6}\right)\left(\frac{a'^2}{a^2}-\kappa\right)\right]\right\}$$

$$+\frac{1}{4\pi^2 a^4}\int d\bar\mu(k)\frac{1}{k^3}\left[-\frac{m^4 a^4}{8}-\frac{m^2\lambda\phi^2 a^4}{8}-\frac{\lambda^2\phi^4 a^4}{32}-\frac{3m^2 a^2}{2}\right.$$

$$\times\left(\frac{a'^2}{a^2}+\kappa\right)-6\left(\xi-\frac{1}{6}\right)\frac{\lambda}{8}(\phi^2 a'^2+2\phi\phi'aa'+\kappa\phi^2 a^2)$$

$$\left.+\left(\xi-\frac{1}{6}\right)^2{}^{(1)}H^\eta{}_\eta\frac{a^4}{4}\right], \tag{4.1b}$$

$$\langle T\rangle_d = \frac{1}{4\pi^2 a^4}\int d\mu(k)\frac{1}{k}\left[m^2 a^2+\frac{\lambda\phi^2 a^2}{2}-6\left(\xi-\frac{1}{6}\right)\left(\frac{a''}{a}-\frac{a'^2}{a^2}\right)\right]$$

$$+\frac{1}{4\pi^2 a^4}\int d\bar\mu(k)\frac{1}{k^3}\left\{-\frac{m^4 a^4}{2}-\frac{m^2\lambda\phi^2 a^4}{2}-\frac{\lambda^2\phi^4 a^4}{8}-\left(\xi-\frac{1}{6}\right)\right.$$

$$\times\left[3m^2 a^2\left(\frac{a''}{a}+\kappa\right)+\frac{\lambda}{4}(6\phi\phi''a^2+6\phi'^2 a^2+12\phi\phi'aa'\right.$$

$$\left.\left.+6\phi^2 aa''+6\kappa\phi^2 a^2)\right]+\left(\xi-\frac{1}{6}\right)^2{}^{(1)}H\frac{a^4}{4}\right\}, \tag{4.1c}$$

with

$$\int d\bar\mu(k)\equiv\int_\varepsilon^{+\infty}dk\,k^2 \quad\text{if }\kappa=0,-1,$$

$$\equiv\sum_{k=1}^{+\infty}k^2 \qquad\text{if }\kappa=+1.$$

Here ε is an arbitrary lower limit cutoff and

$$^{(1)}H_{\mu\nu}=2R_{;\mu\nu}-2g_{\mu\nu}\Box R-\frac{1}{2}g_{\mu\nu}R^2+2RR_{\mu\nu}. \tag{4.2a}$$

In a RW spacetime it has the componenets

$$^{(1)}H^\eta{}_\eta=-\frac{36a'a'''}{a^6}+\frac{72a'^2 a''}{a^7}+\frac{18a''^2}{a^6}+\frac{36\kappa a'^2}{a^6}-\frac{18\kappa^2}{a^4}, \tag{4.2b}$$

$$^{(1)}H=-\frac{36a''''}{a^5}+\frac{144a'a'''}{a^6}-\frac{216a'^2 a''}{a^7}+\frac{108a''^2}{a^6}+\frac{72\kappa a''}{a^5}-\frac{72\kappa a'^2}{a^6}. \tag{4.2c}$$

The renormalized energy–momentum tensor is then computed by subtracting and adding the quantity $\langle T_{\mu\nu}\rangle_d$ to Eq. (3.2b), with the result that

$$\langle T_{\mu\nu}\rangle_R \equiv \langle T_{\mu\nu}\rangle_n + \langle T_{\mu\nu}\rangle_{an}, \tag{4.3a}$$

$$\langle T_{\mu\nu}\rangle_n \equiv \langle T_{\mu\nu}\rangle_B - \langle T_{\mu\nu}\rangle_d, \tag{4.3b}$$

$$\langle T_{\mu\nu}\rangle_{an} \equiv \langle T_{\mu\nu}\rangle_d - \langle T_{\mu\nu}\rangle_{ad}. \tag{4.3c}$$

In general $\langle T_{\mu\nu}\rangle_n$ must be computed numerically while $\langle T_{\mu\nu}\rangle_{an}$ can always be computed analytically. The result is

$$\langle \psi^2\rangle_{an} = \langle \psi^2\rangle_{an}^F - \frac{\lambda\phi^2}{16\pi^2}\left\{1 - \left[\log\left(\frac{2\varepsilon}{a\mu}\right) - \frac{\kappa(\kappa+1)}{2}(\log\varepsilon + \gamma)\right]\right\}, \tag{4.4a}$$

$$\begin{aligned}
\langle T^\eta{}_\eta\rangle_{an} = \langle T^\eta{}_\eta\rangle_{an}^F &+ \frac{\lambda}{4\pi^2}\left\{-\frac{\phi^2}{48}\frac{\kappa(\kappa+1)}{2a^2} - \frac{\phi^2}{16}\left(m^2 + \frac{\lambda\phi^2}{2}\right)\right.\\
&\left.+ \frac{\phi^2}{8}\left(m^2 + \frac{\lambda\phi^2}{4}\right)\left[\log\left(\frac{2\varepsilon}{a\mu}\right) - \frac{\kappa(\kappa+1)}{2}(\log\varepsilon + \gamma)\right]\right\}\\
&+ \frac{(6\xi - 1)\lambda}{16\pi^2 a^4}\left(a^2\kappa\frac{\phi^2}{2} + a'^2\frac{\phi^2}{2} + aa'\phi\phi'\right)\left[\log\left(\frac{2\varepsilon}{a\mu}\right)\right.\\
&\left.- \frac{\kappa(\kappa+1)}{2}(\log\varepsilon + \gamma)\right] - \frac{\lambda}{96\pi^2 a^4}\left(a'^2\frac{\phi^2}{2} + aa'\phi\phi'\right)\\
&- \frac{(6\xi - 1)\lambda}{16\pi^2 a^4}\left(a^2\kappa\frac{\phi^2}{2} + a'^2\phi^2 + aa'\phi\phi'\right), \tag{4.4b}
\end{aligned}$$

$$\begin{aligned}
\langle T\rangle_{an} = \langle T\rangle_{an}^F &+ \frac{\lambda}{4\pi^2}\left\{-\frac{\phi^2}{48}\frac{\kappa(\kappa+1)}{a^2} - \frac{\phi^2}{4}\left(\frac{3m^2}{2} + \frac{\lambda\phi^2}{2}\right)\right.\\
&\left.+ \frac{\phi^2}{4}\left(2m^2 + \frac{\lambda\phi^2}{2}\right)\left[\log\left(\frac{2\varepsilon}{a\mu}\right) - \frac{\kappa(\kappa+1)}{2}(\log\varepsilon + \gamma)\right]\right\}\\
&+ \frac{(6\xi - 1)\lambda}{16\pi^2 a^3}(a\kappa\phi^2 + a''\phi^2 + 2a'\phi\phi' + a\phi'^2 + a\phi\phi'')\left[\log\left(\frac{2\varepsilon}{a\mu}\right)\right.\\
&\left.- \frac{\kappa(\kappa+1)}{2}(\log\varepsilon + \gamma)\right] - \frac{\lambda}{96\pi^2 a^4}\left(3a^4\lambda\frac{\phi^4}{4} + a^2\phi'^2 + a^2\phi\phi''\right.\\
&\left.+ aa''\phi^2 + 2aa'\phi\phi'\right) - \frac{(6\xi - 1)\lambda}{16\pi^2 a^4}\left(3a^2\kappa\frac{\phi^2}{2} + a'^2\frac{\phi^2}{2} + 4aa'\phi\phi'\right.\\
&\left.+ 2aa''\phi^2 + a^2\phi'^2 + a^2\phi\phi''\right), \tag{4.4c}
\end{aligned}$$

with

$$\langle\psi^2\rangle^F_{an} = -\frac{a''}{48\pi^2 a^3} + \frac{1}{4\pi^2}\left\{-\frac{m^2}{4} - \frac{\kappa(\kappa+1)}{24a^2} + \frac{m^2}{2}\left[\log\left(\frac{2\varepsilon}{a\mu}\right)\right.\right.$$
$$\left.\left. - \frac{\kappa(\kappa+1)}{2}(\log\varepsilon + \gamma)\right]\right\} - \frac{(\xi-1/6)R}{8\pi^2}\left\{1 - \left[\log\left(\frac{2\varepsilon}{a\mu}\right)\right.\right.$$
$$\left.\left. - \frac{\kappa(\kappa+1)}{2}(\log\varepsilon + \gamma)\right]\right\}, \tag{4.5a}$$

$$\langle T^\eta_{\ \eta}\rangle^F_{an} = \frac{1}{2880\pi^2}\left[-\frac{1}{6}{}^{(1)}H^\eta_{\ \eta} + {}^{(3)}H^\eta_{\ \eta} - \frac{3\kappa(\kappa+1)}{a^4}\right] + \frac{m^2}{288\pi^2}G^\eta_{\ \eta}$$
$$- \frac{m^2\kappa(\kappa-1)}{192\pi^2 a^2} - \frac{m^4}{64\pi^2}\left[\frac{1}{2} + 2\,\log\left(\frac{\mu a}{2\varepsilon}\right) + \kappa(\kappa+1)(\gamma + \log\varepsilon)\right]$$
$$+ \left(\xi - \frac{1}{6}\right)\left[\frac{{}^{(1)}H^\eta_{\ \eta}}{288\pi^2} + \frac{\kappa(\kappa-1)}{32\pi^2 a^4}\left(1 + \frac{a'^2}{a^2}\right) + \frac{m^2}{16\pi^2}G^\eta_{\ \eta}\right.$$
$$\times \left.\left(3 + 2\,\log\left(\frac{\mu a}{2\varepsilon}\right) + \kappa(\kappa+1)(\gamma + \log\varepsilon)\right) + \frac{3\kappa m^2}{8\pi^2 a^2}\right]$$
$$+ \left(\xi - \frac{1}{6}\right)^2\left[\frac{{}^{(1)}H^\eta_{\ \eta}}{32\pi^2}\left(2 + 2\,\log\left(\frac{\mu a}{2\varepsilon}\right) + \kappa(\kappa+1)(\gamma + \log\varepsilon)\right)\right.$$
$$\left. - \frac{9}{4\pi^2}\left(\frac{a'^2 a''}{a^7} + \frac{\kappa a'^2}{a^6}\right)\right], \tag{4.5b}$$

$$\langle T\rangle^F_{an} = \frac{1}{2880\pi^2}\left[-\frac{1}{6}{}^{(1)}H + {}^{(3)}H\right] + \frac{m^2}{288\pi^2}G - \frac{m^2\kappa(\kappa-1)}{96\pi^2 a^2}$$
$$- \frac{m^4}{16\pi^2}\left[1 + 2\,\log\left(\frac{\mu a}{2\varepsilon}\right) + \kappa(\kappa+1)(\gamma + \log\varepsilon)\right] + \left(\xi - \frac{1}{6}\right)$$
$$\times \left[\frac{{}^{(1)}H}{288\pi^2} + \frac{\kappa(\kappa-1)}{16\pi^2 a^4}\left(\frac{a''}{a} - \frac{a'^2}{a^2}\right) + \frac{m^2}{16\pi^2}G\left(3 + 2\,\log\left(\frac{\mu a}{2\varepsilon}\right)\right.\right.$$
$$\left.\left. + \kappa(\kappa+1)(\gamma + \log\varepsilon)\right) + \frac{3\kappa m^2}{8\pi^2 a^2} - \frac{3m^2 a'^2}{8\pi^2 a^4}\right] + \left(\xi - \frac{1}{6}\right)^2$$
$$\times \left[\frac{{}^{(1)}H}{32\pi^2}\left(2 + 2\,\log\left(\frac{\mu a}{2\varepsilon}\right) + \kappa(\kappa+1)(\gamma + \log\varepsilon)\right)\right.$$
$$\left. - \frac{9}{8\pi^2}\left(\frac{4a'a'''}{a^6} - \frac{10a'^2 a''}{a^7} + \frac{3a''^2}{a^6} + \frac{4\kappa a''}{a^5} - \frac{6\kappa a'^2}{a^6} + \frac{\kappa^2}{a^4}\right)\right]. \tag{4.5c}$$

Here $G_{\mu\nu}$ is the Einstein tensor with components

$$G^{\eta}{}_{\eta} = -\frac{3a'^2}{a^4} - \frac{3\kappa}{a^2}, \tag{4.6a}$$

$$G = -\frac{6a''}{a^3} - \frac{6\kappa}{a^2} = -R, \tag{4.6b}$$

and $^{(3)}H_{\mu\nu}$ is the tensor

$$^{(3)}H_{\mu\nu} = R_{\mu}{}^{\rho}R_{\rho\nu} - \frac{2}{3}RR_{\mu\nu} - \frac{1}{2}R_{\rho\sigma}R^{\rho\sigma}g_{\mu\nu} + \frac{1}{4}R^2 g_{\mu\nu}, \tag{4.7a}$$

with components

$$^{(3)}H^{\eta}{}_{\eta} = \frac{3a'^4}{a^8} + \frac{6\kappa a'^2}{a^6} + \frac{3\kappa^2}{a^4}, \tag{4.7b}$$

$$^{(3)}H = \frac{12a'^2 a''}{a^7} - \frac{12a'^4}{a^8} + \frac{12\kappa a''}{a^5} - \frac{12\kappa a'^2}{a^6}. \tag{4.7c}$$

For a massive field $\mu = m$, while for a massless field μ is an arbitrary mass scale. However, in the massless case each of the terms in $\langle T_{\mu\nu}\rangle_{an}$ which contains a factor of $\log \mu$ has as a coefficient a multiple of the tensor $^{(1)}H_{\mu\nu}$, which comes from an R^2 term in the gravitational Lagrangian. Thus, the terms containing $\log \mu$ simply correspond to a finite renormalization of the coefficient of the R^2 term in the gravitational action.

Note that if $\kappa = +1$, $\langle \psi^2 \rangle_d$ and $\langle T_{\mu\nu}\rangle_d$ contain a sum over k, while $\langle \psi^2 \rangle_{ad}$ and $\langle T_{\mu\nu}\rangle_{ad}$ contain an integral over k. Thus, either the integral must be converted to a sum or the sum to an integral. We have converted the sum to an integral using the Plana sum formula (Ghika and Visinescu, 1978; Lindelof, 1905; Shen *et al.*, 1985; Whittaker and Watson, 1927). This formula is

$$\sum_{n=m}^{+\infty} f(n) = \frac{1}{2}f(m) + \int_{m}^{+\infty} dx\, f(x) + i\int_{0}^{+\infty} \frac{dt}{e^{2\pi t} - 1}[f(m+it) - f(m-it)]. \tag{4.8}$$

Because of the way $\langle T_{\mu\nu}\rangle_d$ is defined, the third term in the Plana sum formula can be computed exactly. In the traditional form of adiabatic regularization one would convert the integral in the adiabatic counterterms to a sum using the Plana sum formula and then substitute the result into Eq. (3.2b). However, if this is done then, for a massive field, it is not possible to compute the third term in the Plana sum formula analytically. Thus, the computation of the renormalized energy–momentum tensor is simplified by our method in the $\kappa = +1$ case. The same simplification would occur if one was using compact spatial sections for $\kappa = 0$ or $\kappa = -1$ RW spacetimes.

5. CONSERVATION OF THE ENERGY–MOMENTUM TENSOR

In this section we show that the renormalized energy–momentum tensor for the $\lambda\phi^4$ theory (at one-loop) in a RW spacetime is covariantly conserved. In a RW spacetime there is only one nontrivial conservation equation which is

$$T^{\eta}{}_{\eta,\eta} + \frac{4a'}{a}T^{\eta}{}_{\eta} - \frac{a'}{a}T = 0. \tag{5.1}$$

We have shown that the energy–momentum tensor for the field can be divided into a classical and a quantum part. First consider the classical part which in a RW spacetime has the components (3.3c) and (3.3d). Substituting into Eq. (5.1) and using (3.3a) we find

$$\left\langle T^{\eta}{}_{\eta}\right\rangle^{C}_{R,\eta} + \frac{4a'}{a}\left\langle T^{\eta}{}_{\eta}\right\rangle^{C}_{R} - \frac{a'}{a}\left\langle T\right\rangle^{C}_{R} = -\frac{\lambda}{2}\phi\phi'\langle\psi^2\rangle_R. \tag{5.2}$$

Thus, the classical energy–momentum tensor is conserved only if there is no quantum correction to the effective mass of the mean field.

The renormalized energy–momentum tensor for the quantum part consists of the difference between the bare contribution and the adiabatic counterterms. The components of the bare part are given in Eqs. (3.8a) and (3.8b). If they are substituted into Eq. (5.1) and the Eq. (3.3b) is used then one finds

$$\left\langle T^{\eta}{}_{\eta}\right\rangle^{Q}_{B,\eta} + \frac{4a'}{a}\left\langle T^{\eta}{}_{\eta}\right\rangle^{Q}_{B} - \frac{a'}{a}\left\langle T\right\rangle^{Q}_{B} = \frac{\lambda}{2}\phi\phi'\langle\psi^2\rangle_B. \tag{5.3}$$

If one substitutes the adiabatic counterterms (3.12b) and (3.12c) into Eq. (5.1) and compares the result with Eq. (3.2a), one finds

$$\left\langle T^{\eta}{}_{\eta}\right\rangle_{ad,\eta} + \frac{4a'}{a}\left\langle T^{\eta}{}_{\eta}\right\rangle_{ad} - \frac{a'}{a}\left\langle T\right\rangle_{ad} = \frac{\lambda}{2}\phi\phi'\langle\psi^2\rangle_{ad}. \tag{5.4}$$

Combining these results and using (3.2a) and (3.2b), one finds that the total renormalized energy–momentum tensor, classical plus quantum, is conserved. (See Appendix A and Appendix B for the details of the proof.)

One further finds that if the analytic approximation is used in place of $\langle\psi^2\rangle_R$ in the equation for the mean field and if it is used for the quantum energy–momentum tensor, then the (full) analytically approximated energy–momentum tensor is conserved. This means that one can use the analytic approximation to define a consistent set of equations for the mean and the quantum fields, and to be the source in the right-hand-side of Einstein's equation to solve a first approximation to the backreaction problem.

Thus $\langle\psi^2\rangle_{an}$ and $\langle T_{\mu\nu}\rangle_{an}$ can be used in place of $\langle\psi^2\rangle_R$ and $\langle T_{\mu\nu}\rangle_R$ in Eqs. (2.9a), (2.10a), and (2.10b) to obtain an analytic approximation for the system. For $\kappa = 0, -1$ RW spacetimes the terms being approximated contain the arbitrary constant ε. This means the approximation is not unique unless the coefficients of

the log ε terms vanish. It is important to note that this is only true when using $\langle \psi^2 \rangle_{an}$ and $\langle T_{\mu\nu} \rangle_{an}$ as an analytic approximation. The ε dependent terms do not appear in the exact renormalized expressions for these quantities.

From the DeWitt-Schwinger expansion (Christensen, 1976, 1978) it is known that for a free quantum field in the large mass limit $\langle \psi^2 \rangle_R$ and $\langle T_{\mu\nu} \rangle_R$ have leading order terms proportional to $1/m^2$. Thus, the analytic approximations for these quantities are not good approximations in this limit. Previous numerical work (Anderson, 1985, 1986) indicates that the relevant condition is likely to be $ma \ll 1$. The analytically approximated quantities are also local in the sense that they depend on the scale factor and its derivatives at a given time η. Therefore, they cannot accurately describe particle production effects which are inherently nonlocal. However, for massless fields they should allow one to estimate how important vacuum polarization effects are and how they qualitatively modify the evolution of the system.

6. SUMMARY

We have used adiabatic regularization to renormalize a scalar field theory with a quartic self-coupling (of the form $\lambda\phi^4$) in an arbitrary RW spacetime. We have found that the energy–momentum tensor can be naturally split into two parts, a "classical" contribution (which corresponds to the energy–momentum tensor of a classical scalar field with a quartic interaction in a RW spacetime) and a "quantum" piece (which corresponds to the energy–momentum tensor of a free quantum scalar field with the time dependent mass $m^2 + \frac{\lambda\phi^2}{2}$). We have displayed the renormalization counterterms for both the energy–momentum tensor and the contribution of the quantum fluctuations to the effective mass of the mean field at one-loop order. We have directly checked to see if the energy momentum tensor is covariantly conserved and found that while the entire tensor is conserved, its classical and quantum contributions are not separately conserved.

By using a variant on the adiabatic regularization method we have derived analytic approximations for the energy–momentum tensor and the contribution of the quantum fluctuations to the effective mass of the mean field. We have shown that the analytically approximated energy–momentum tensor is covariantly conserved. Thus the analytic approximations can be used in a self-consistent way to find approximate solutions to the mean field and backreaction equations. The approximations can provide a useful tool for learning about vacuum polarization effects for massless fields. However, they are not useful for massive fields in the large mass limit. They do not give any significant amount of information about particle production, which is a nonlocal phenomenon.

The approximations could be of use in reheating calculations, in particular the transition from the slow-roll dynamics of the inflaton field to the oscillatory behavior around the minimum of the effective potential, (which is believed to produce

particles of lighter masses), and in the context of relativistic heavy ion collision as a way of estimating the physical energy density and pressure of the vacuum and thermal excitations. The advantage of this approximation is that one obtains analytic expressions for the quantum piece of the energy–momentum tensor, without need to solve exactly the mode equation (which is the most difficult part to implement in numeric computations). In this way, it is relatively easy to study the backreaction problem of the full system (mean field, quantum fluctuations, and gravitational field).

We believe that this program can be carried forward and improved easily. We plan to extend the present approach and approximations to two-loop order and to the $1/N$ expansion. We also plan to perform some specific calculations with applications to reheating and early density perturbations.

APPENDIX A: CONSERVATION OF THE BARE ENERGY–MOMENTUM TENSOR

In this Appendix we show that the bare energy–momentum tensor for the $\lambda\phi^4$ theory (at one-loop) is covariantly conserved. We start by looking at the equations of motion

$$(\Box + m^2 + \xi R)\phi + \frac{\lambda}{3!}\phi^3 + \frac{\lambda}{2}\phi\langle\psi^2\rangle = 0, \tag{A.1a}$$

$$(\Box + m^2 + \xi R)\psi + \frac{\lambda}{2}\phi^2\psi = 0. \tag{A.1b}$$

We assume that the mean field is homogeneous $\phi = \phi(\eta)$ and write in conformal time (primes are time derivatives with respect to conformal time)

$$\phi'' + 2\frac{a'}{a}\phi' + a^2\left(m^2 + \xi R + \frac{\lambda}{3!}\phi^2 + \frac{\lambda}{2}\langle\psi^2\rangle\right)\phi = 0, \tag{A.2a}$$

$$\psi'' + 2\frac{a'}{a}\psi' - \Delta^{(3)}\psi + a^2\left(m^2 + \xi R + \frac{\lambda}{2}\phi\right)\psi = 0. \tag{A.2b}$$

The bare energy–momentum tensor is given by

$$\begin{aligned}
\langle T_{\mu\nu}(\phi, \psi)\rangle_B &= (1 - 2\xi_B)\partial_\mu\phi\partial_\nu\phi + (2\xi_B - 1/2)g_{\mu\nu}\partial_\tau\phi\partial^\tau\phi - 2\xi_B\phi\nabla_\mu\nabla_\nu\phi \\
&\quad + 2\xi_B g_{\mu\nu}\phi\Box\phi - \xi_B G_{\mu\nu}\phi^2 + \frac{m_B^2}{2}g_{\mu\nu}\phi^2 + \frac{\lambda_B}{4!}g_{\mu\nu}\phi^4 \\
&\quad + (1 - 2\xi_B)\langle\partial_\mu\psi\partial_\nu\psi\rangle + (2\xi_B - 1/2)g_{\mu\nu}\langle\partial_\tau\psi\partial^\tau\psi\rangle \\
&\quad - 2\xi_B\langle\psi\nabla_\mu\nabla_\nu\psi\rangle + 2\xi_B g_{\mu\nu}\langle\psi\Box\psi\rangle - \xi_B G_{\mu\nu}\langle\psi^2\rangle \\
&\quad + \frac{m_B^2}{2}g_{\mu\nu}\langle\psi^2\rangle + \frac{\lambda_B}{4!}g_{\mu\nu}6\phi^2\langle\psi^2\rangle, \tag{A.3a}
\end{aligned}$$

with the first two lines corresponding to the classical piece of the energy–momentum tensor and the last two lines to the quantum contribution, that is

$$\langle T_{\mu\nu}(\phi, \psi)\rangle_B^C \overset{\text{def}}{=} (1 - 2\xi_B)\partial_\mu\phi\partial_\nu\phi + (2\xi_B - 1/2)g_{\mu\nu}\partial_\tau\phi\partial^\tau\phi - 2\xi_B\phi\nabla_\mu\nabla_\nu\phi$$

$$+ 2\xi_B g_{\mu\nu}\phi\Box\phi - \xi_B G_{\mu\nu}\phi^2 + \frac{m_B^2}{2}g_{\mu\nu}\phi^2 + \frac{\lambda_B}{4!}g_{\mu\nu}\phi^4, \quad \text{(A.3b)}$$

$$\langle T_{\mu\nu}(\phi, \psi)\rangle_B^Q \overset{\text{def}}{=} (1 - 2\xi_B)\langle\partial_\mu\psi\partial_\nu\psi\rangle + (2\xi_B - 1/2)g_{\mu\nu}\langle\partial_\tau\psi\partial^\tau\psi\rangle$$

$$- 2\xi_B\langle\psi\nabla_\mu\nabla_\nu\psi\rangle + 2\xi_B g_{\mu\nu}\langle\psi\Box\psi\rangle - \xi_B G_{\mu\nu}\langle\psi^2\rangle$$

$$+ \frac{m_B^2}{2}g_{\mu\nu}\langle\psi^2\rangle + \frac{\lambda_B}{4}g_{\mu\nu}\phi^2\langle\psi^2\rangle. \quad \text{(A.3c)}$$

The $\eta\eta$ component is given by

$$\langle T_{\eta\eta}(\phi, \psi)\rangle_B = \frac{1}{2}\phi'\phi' + 6\xi_B\frac{a'}{a}\phi\phi' + 3\xi_B\left(\frac{a'^2}{a^2} + \kappa\right)\phi^2 + \frac{a^2 m_B^2}{2}\phi^2 + \frac{a^2\lambda_B}{4!}\phi^4$$

$$+ \frac{1}{2}\langle\psi'\psi'\rangle - \frac{1}{2}\langle\psi\Delta^{(3)}\psi\rangle + 6\xi_B\frac{a'}{a}\langle\psi\psi'\rangle + 3\xi_B\left(\frac{a'^2}{a^2} + \kappa\right)\langle\psi^2\rangle$$

$$+ \frac{a^2 m_B^2}{2}\langle\psi^2\rangle + \frac{a^2\lambda_B}{4}\phi^2\langle\psi^2\rangle, \quad \text{(A.4a)}$$

and the trace

$$\langle T(\phi, \psi)\rangle_B = (6\xi_B - 1)\frac{1}{a^2}\phi'\phi' + \frac{6}{a^2}\xi_B\phi\phi'' + \frac{12}{a^2}\xi_B\frac{a'}{a}\phi\phi'$$

$$+ \xi_B R\phi^2 + 2m_B^2\phi^2 + \frac{\lambda_B}{3!}\phi^4 + (6\xi_B - 1)\frac{1}{a^2}\langle\psi'\psi'\rangle$$

$$- \frac{1}{a^2}\langle\psi\Delta^{(3)}\psi\rangle + \frac{6}{a^2}\xi_B\langle\psi\psi''\rangle + \frac{12}{a^2}\xi_B\frac{a'}{a}\langle\psi\psi'\rangle$$

$$+ \xi_B R\langle\psi^2\rangle + 2m_B^2\langle\psi^2\rangle + \lambda_B\phi^2\langle\psi^2\rangle. \quad \text{(A.4b)}$$

In order to show that the bare energy–momentum tensor is covariantly conserved, we must prove that

$$\partial_\eta\langle T^\eta{}_\eta(\phi, \psi)\rangle_B + \frac{a'}{a}\left[4\langle T^\eta{}_\eta(\phi, \psi)\rangle_B - \langle T(\phi, \psi)\rangle_B\right] = 0. \quad \text{(A.5)}$$

In order to make the proof a little bit easier, let us introduce the following notation

$$\langle T^\eta{}_\eta(\phi, \psi)\rangle_B \overset{\text{def}}{=} \frac{1}{a^2}\mathcal{K}_\eta, \quad \text{and} \quad \langle T(\phi, \psi)\rangle_B \overset{\text{def}}{=} \frac{1}{a^2}\mathcal{K}_T. \quad \text{(A.6)}$$

The conservation equation now becomes

$$\partial_\eta \mathcal{K}_\eta + \frac{a'}{a}(2\mathcal{K}_\eta - \mathcal{K}_T) = 0, \tag{A.7}$$

with

$$\mathcal{K}_\eta = \frac{1}{2}\phi'\phi' + 6\xi_B \frac{a'}{a}\phi\phi' + 3\xi_B \left(\frac{a'^2}{a^2} + \kappa\right)\phi^2 + \frac{a^2 m_B^2}{2}\phi^2 + \frac{a^2 \lambda_B}{4!}\phi^4$$

$$+ \frac{1}{2}\langle\psi'\psi'\rangle - \frac{1}{2}\langle\psi\Delta^{(3)}\psi\rangle + 6\xi_B \frac{a'}{a}\langle\psi\psi'\rangle + 3\xi_B \left(\frac{a'^2}{a^2} + \kappa\right)\langle\psi^2\rangle$$

$$+ \frac{a^2 m_B^2}{2}\langle\psi^2\rangle + \frac{a^2 \lambda_B}{4}\phi^2\langle\psi^2\rangle, \tag{A.8a}$$

$$\mathcal{K}_T = (6\xi_B - 1)\phi'\phi' + 6\xi_B \phi\phi'' + 12\xi_B \frac{a'}{a}\phi\phi' + \xi_B a^2 R\phi^2 + 2a^2 m_B^2 \phi^2$$

$$+ a^2 \frac{\lambda_B}{3!}\phi^4 + (6\xi_B - 1)\langle\psi'\psi'\rangle - \langle\psi\Delta^{(3)}\psi\rangle + 6\xi_B\langle\psi\psi''\rangle$$

$$+ 12\xi_B \frac{a'}{a}\langle\psi\psi'\rangle + \xi_B a^2 R\langle\psi^2\rangle + 2a^2 m_B^2\langle\psi^2\rangle + a^2 \lambda_B \phi^2\langle\psi^2\rangle. \tag{A.8b}$$

In order to present a more explicit proof, we treat the classical and quantum pieces separately. We start with the classical terms[9]

$$\partial_\eta \mathcal{K}_\eta^C = \phi'\phi'' + 6\xi \frac{a'}{a}(\phi'\phi' + \phi\phi'') + 6\xi\left(\frac{a''}{a} - \frac{a'^2}{a^2}\right)\phi\phi'$$

$$+ 6\xi\left(\kappa + \frac{a'^2}{a^2}\right)\phi\phi' + 6\xi \frac{a'}{a}\left(\frac{a''}{a} - \frac{a'^2}{a^2}\right)\phi^2 + aa'm^2\phi^2$$

$$+ a^2 m^2\phi\phi' + \frac{aa'\lambda}{12}\phi^4 + \frac{a^2 \lambda}{3!}\phi^3\phi', \tag{A.8c}$$

$$2\mathcal{K}_\eta^C - \mathcal{K}_T^C = \phi'\phi' + 12\xi \frac{a'}{a}\phi\phi' + 6\xi\left(\frac{a'^2}{a^2} + \kappa\right)\phi^2$$

$$+ a^2 m^2\phi^2 + \frac{a^2 \lambda}{12}\phi^4 + (1 - 6\xi)\phi'\phi' - 6\xi\phi\phi'' - 12\xi \frac{a'}{a}\phi\phi'$$

$$- \xi_B a^2 R\phi^2 - 2a^2 m_B^2 \phi^2 - a^2 \frac{\lambda_B}{3!}\phi^4$$

$$= (2 - 6\xi)\phi'\phi' - 6\xi\phi\phi'' + 6\xi\left(\frac{a'^2}{a^2} - \frac{a''}{a}\right)\phi^2 - a^2 m^2\phi^2 - \frac{a^2 \lambda}{12}\phi^4. \tag{A.8d}$$

[9] We have dropped the bare subscript from the coupling constants m, ξ, and λ.

We can conclude then [by making use of the equation of motion for $\phi(\eta)$, Eq. (A.2a), in the term $\phi\phi''$], that

$$\partial_\eta \mathcal{K}_\eta^C + \frac{a'}{a}\left(2\mathcal{K}_\eta^C - \mathcal{K}_T^C\right) = -a^2\frac{\lambda}{2}\phi\phi'\langle\psi^2\rangle,$$

$$\partial_\eta \langle T^\eta{}_\eta\rangle^C + \frac{a'}{a}\left(4\langle T^\eta{}_\eta\rangle^C - \langle T\rangle^C\right) = -\frac{\lambda}{2}\phi\phi'\langle\psi^2\rangle. \tag{A.9}$$

The proof for the quantum contribution to the energy–momentum tensor goes along the same lines. By defining[10]

$$\mathcal{K}_\eta^Q = \frac{1}{2}\langle\psi'\psi'\rangle - \frac{1}{2}\langle\psi\Delta^{(3)}\psi\rangle + 6\xi_B\frac{a'}{a}\langle\psi\psi'\rangle + 3\xi_B\left(\frac{a'^2}{a^2}+\kappa\right)\langle\psi^2\rangle$$

$$+ \frac{a^2 m_B^2}{2}\langle\psi^2\rangle + \frac{a^2\lambda_B}{4}\phi^2\langle\psi^2\rangle, \tag{A.10a}$$

$$\mathcal{K}_T^Q = (6\xi_B - 1)\langle\psi'\psi'\rangle - \langle\psi\Delta^{(3)}\psi\rangle + 6\xi_B\langle\psi\psi''\rangle + 12\xi_B\frac{a'}{a}\langle\psi\psi'\rangle$$

$$+ \xi_B a^2 R\langle\psi^2\rangle + 2a^2 m_B^2\langle\psi^2\rangle + a^2\lambda_B\phi^2\langle\psi^2\rangle, \tag{A.10b}$$

we have then

$$\partial_\eta\mathcal{K}_\eta^Q = \langle\psi'\psi''\rangle - \langle\psi'\Delta^{(3)}\psi\rangle + 6\xi\frac{a'}{a}\langle\psi'\psi' + \psi\psi''\rangle$$

$$+ 6\xi\left(\frac{a''}{a} - \frac{a'^2}{a^2}\right)\langle\psi\psi'\rangle + 6\xi\left(\frac{a'^2}{a^2}+\kappa\right)\langle\psi\psi'\rangle$$

$$+ 6\xi\frac{a'}{a}\left(\frac{a''}{a} - \frac{a'^2}{a^2}\right)\langle\psi\psi'\rangle + aa'm^2\langle\psi^2\rangle + a^2 m^2\langle\psi\psi'\rangle$$

$$+ \frac{a^2\lambda}{2}\phi^2\langle\psi\psi'\rangle + \frac{a^2\lambda}{2}\phi\phi'\langle\psi^2\rangle + \frac{aa'\lambda}{2}\phi^2\langle\psi^2\rangle, \tag{A.10c}$$

$$2\mathcal{K}_\eta^Q - \mathcal{K}_T^Q = (2 - 6\xi)\langle\psi'\psi'\rangle - 6\xi\langle\psi\psi''\rangle + 6\xi\left(\frac{a'^2}{a^2} - \frac{a''}{a}\right)\langle\psi^2\rangle$$

$$- a^2 m^2\langle\psi^2\rangle - \frac{a^2\lambda}{2}\phi^2\langle\psi^2\rangle. \tag{A.10d}$$

---

[10] As the quantum contribution to the full energy–momentum tensor is already one-loop, we can consider all the parameters, m, ξ, and λ to be the renormalized ones.

We can conclude then [by making use of the equation of motion for ψ, Eq. (A.2b), in the term $\psi\psi''$], that

$$\partial_\eta \mathcal{K}_n^Q + \frac{a'}{a}\left(2\mathcal{K}_\eta^Q - \mathcal{K}_T^Q\right) = a^2 \frac{\lambda}{2}\phi\phi'\langle\psi^2\rangle,$$

$$\partial_\eta \langle T^\eta{}_\eta\rangle^Q + \frac{a'}{a}\left(4\langle T^\eta{}_\eta\rangle^Q - \langle T\rangle^Q\right) = \frac{\lambda}{2}\phi\phi'\langle\psi^2\rangle. \qquad (A.11)$$

If we consider the sum of both contributions [Eqs. (A.9) and (A.11)], it is easy to see that

$$\partial_\eta \langle T^\eta{}_\eta(\phi, \psi)\rangle_B + \frac{a'}{a}\left[4\langle T^\eta{}_\eta(\phi, \psi)\rangle_B - \langle T(\phi, \psi)\rangle_B\right] = 0. \qquad (A.12)$$

One important point that needs to be mentioned, although it should be clear from the previous proofs, is that neither the classical, nor the quantum piece of the energy–momentum tensor is conserved by itself, as has been shown above.

APPENDIX B: CONSERVATION OF THE RENORMALIZED ENERGY–MOMENTUM TENSOR

The first question that needs to be addressed is how to define the renormalized energy–momentum tensor. In order to answer this question, we point out at this stage that the divergences of $\langle\psi^2\rangle_B$ can all be absorbed in the bare coupling constants m_B^2, ξ_B, and λ_B, as we have shown in a previous Section, after defining

$$\langle\psi^2\rangle_R = \langle\psi^2\rangle_B - \langle\psi^2\rangle_{ad}. \qquad (B.1)$$

This scheme defines the renormalized parameters m_R^2, ξ_R, and λ_R. We can then write the renormalized equation of motion for the mean field in terms of these parameters to obtain

$$\phi'' = -\frac{2a'}{a}\phi' - a^2\left(m_R^2 + \xi_R R + \frac{\lambda_R}{2}\phi^2\right)\phi - \frac{\lambda_R}{2}\phi\langle\psi^2\rangle_R, \qquad (B.2)$$

with $\langle\psi^2\rangle_R$ defined by the previous equation.

The bare classical energy–momentum tensor has ultraviolet divergences because it is given in terms of the bare parameters that are divergent themselves. The renormalized classical energy–momentum tensor is given by

$$\langle T_{\mu\nu}(\phi, \psi)\rangle_R^C \stackrel{\text{def}}{=} (1 - 2\xi_R)\partial_\mu\phi\partial_\nu\phi + (2\xi_R - 1/2)g_{\mu\nu}\partial_\tau\phi\partial^\tau\phi - 2\xi_R\phi\nabla_\mu\nabla_\nu\phi$$

$$+ 2\xi_R g_{\mu\nu}\phi\Box\phi - \xi_R G_{\mu\nu}\phi^2 + \frac{m_R^2}{2}g_{\mu\nu}\phi^2 + \frac{\lambda_R}{4!}g_{\mu\nu}\phi^4, \qquad (B.3)$$

which is a finite quantity. It is easy to show that this tensor is not covariantly conserved, in fact

$$\partial_\eta \langle T^\eta{}_\eta(\phi, \psi)\rangle_R^C + \frac{a'}{a}\left[4\langle T^\eta{}_\eta(\phi, \psi)\rangle_R^C - \langle T(\phi, \psi)\rangle_R^C\right] = -\frac{\lambda_R}{2}\phi\phi'\langle\psi^2\rangle_R. \qquad (B.4)$$

Let us now consider the quantum fluctuations. The equation of motion for ψ is

$$
\begin{aligned}
\psi'' &= -\frac{2a'}{a}\psi' + \Delta^{(3)}\psi - a^2\left(m_B^2 + \xi_B R + \frac{\lambda_B}{2}\phi\right)\psi \\
&= -\frac{2a'}{a}\psi' + \Delta^{(3)}\psi - a^2\left(m_R^2 + \xi_R R + \frac{\lambda_R}{2}\phi\right)\psi,
\end{aligned}
\tag{B.5a}
$$

where the equivalence between these two expressions is valid because we are restricting ourselves to one-loop order, and because the quantum field obeys a free equation of motion. The energy–momentum tensor of the quantum fluctuations is given by

$$
\begin{aligned}
\langle T_{\mu\nu}(\phi,\psi)\rangle_B^Q &= (1-2\xi_R)\langle\partial_\mu\psi\partial_\nu\psi\rangle + (2\xi_R - 1/2)g_{\mu\nu}\langle\partial_\tau\psi\partial^\tau\psi\rangle \\
&\quad - 2\xi_R\langle\psi\nabla_\mu\nabla_\nu\psi\rangle + 2\xi_R g_{\mu\nu}\langle\psi\Box\psi\rangle - \xi_R G_{\mu\nu}\langle\psi^2\rangle \\
&\quad + \frac{m_R^2}{2}g_{\mu\nu}\langle\psi^2\rangle + \frac{\lambda_R}{4}g_{\mu\nu}\phi^2\langle\psi^2\rangle,
\end{aligned}
\tag{B.6}
$$

and it corresponds to the energy–momentum tensor of a quantum scalar field with time dependent mass given by

$$
M_R^2(\eta) \equiv m_R^2 + \frac{\lambda_R}{2}\phi^2(\eta).
\tag{B.7}
$$

We can therefore conclude that this energy–momentum tensor has quartic, quadratic, and logarithmic divergences that are in one to one correspondence with the divergences of the corresponding adiabatic order four energy–momentum tensor of a quantum scalar field with curvature coupling ξ_R and mass $M_R^2(\eta)$. We know that the calculation presented by Bunch (1980), or by Anderson and Parker (1987) is only valid for time independent masses. At this moment we do not need to worry about the details of the specific fourth order adiabatic expansion. Let us define the renormalized quantum energy–momentum tensor as

$$
\langle T_{\mu\nu}(\phi,\psi)\rangle_R^Q \overset{\text{def}}{=} \langle T_{\mu\nu}(\phi,\psi)\rangle_B^Q - \langle T_{\mu\nu}(\phi,\psi)\rangle_{ad}^Q.
\tag{B.8}
$$

From previous considerations it is easy to see that

$$
\partial_\eta\langle T^\eta{}_\eta(\phi,\psi)\rangle_B^Q + \frac{a'}{a}\left[4\langle T^\eta{}_\eta(\phi,\psi)\rangle_B^Q - \langle T(\phi,\psi)\rangle_B^Q\right] = \frac{\lambda_R}{2}\phi\phi'\langle\psi^2\rangle_B,
$$

$$
\partial_\eta\langle T^\eta{}_\eta(\phi,\psi)\rangle_{ad}^Q + \frac{a'}{a}\left[4\langle T^\eta{}_\eta(\phi,\psi)\rangle_{ad}^Q - \langle T(\phi,\psi)\rangle_{ad}^Q\right] = \frac{\lambda_R}{2}\phi\phi'\langle\psi^2\rangle_{ad},
$$

$$
\partial_\eta\langle T^\eta{}_\eta(\phi,\psi)\rangle_R^Q + \frac{a'}{a}\left[4\langle T^\eta{}_\eta(\phi,\psi)\rangle_R^Q - \langle T(\phi,\psi)\rangle_R^Q\right] = \frac{\lambda_R}{2}\phi\phi'\langle\psi^2\rangle_R,
\tag{B.9}
$$

where in the last equation $\langle\psi^2\rangle_{ad}$ means this adiabatic quantity of order two. This is necessary to match adiabatic orders properly, as the quantity $\phi(\eta)\phi(\eta)'$ is already

of adiabatic order three. The presence of the adiabatic order four term on the right-hand-side would make it impossible to cancel out the adiabatic order seven terms that this procedure will generate (as the left-hand-side is of adiabatic order five). In fact, it is not very hard to show that indeed, this last equation is not only approximately valid, but that it is an identity.

It is now time to put all these pieces together and define the renormalized energy–momentum tensor for the full system (mean field and fluctuations). We define

$$
\langle T_{\mu\nu}(\phi,\psi)\rangle_R = \langle T_{\mu\nu}(\phi,\psi)\rangle_R^C + \langle T_{\mu\nu}(\phi,\psi)\rangle_R^Q
$$
$$
= \langle T_{\mu\nu}(\phi,\psi)\rangle_R^C + \langle T_{\mu\nu}(\phi,\psi)\rangle_B^Q - \langle T_{\mu\nu}(\phi,\psi)\rangle_{ad}^Q. \quad \text{(B.10)}
$$

The important question that remains to be answered at this stage is the following: is $\langle T_{\mu\nu}(\phi,\psi)\rangle_R$ so defined a covariantly conserved energy–momentum tensor? The answer is given by the following identities

$$
\partial_\eta \langle T^\eta_{\ \eta}(\phi,\psi)\rangle_R^C + \frac{a'}{a}\Big[4\langle T^\eta_{\ \eta}(\phi,\psi)\rangle_R^C - \langle T(\phi,\psi)\rangle_R^C\Big] = -\frac{\lambda_R}{2}\phi\phi'\langle\psi^2\rangle_R,
$$

$$
\partial_\eta \langle T^\eta_{\ \eta}(\phi,\psi)\rangle_B^Q + \frac{a'}{a}\Big[4\langle T^\eta_{\ \eta}(\phi,\psi)\rangle_B^Q - \langle T(\phi,\psi)\rangle_B^Q\Big] = \frac{\lambda_R}{2}\phi\phi'\langle\psi^2\rangle_B,
$$

$$
\partial_\eta \langle T^\eta_{\ \eta}(\phi,\psi)\rangle_{ad}^Q + \frac{a'}{a}\Big[4\langle T^\eta_{\ \eta}(\phi,\psi)\rangle_{ad}^Q - \langle T(\phi,\psi)\rangle_{ad}^Q\Big] = \frac{\lambda_R}{2}\phi\phi'\langle\psi^2\rangle_{ad}. \quad \text{(B.11)}
$$

We conclude therefore that

$$
\partial_\eta \langle T^\eta_{\ \eta}(\phi,\psi)\rangle_R + \frac{a'}{a}\Big[4\langle T^\eta_{\ \eta}(\phi,\psi)\rangle_R - \langle T(\phi,\psi)\rangle_R\Big]
$$
$$
= -\frac{\lambda_R}{2}\phi\phi'\langle\psi^2\rangle_R + \frac{\lambda_R}{2}\phi\phi'\langle\psi^2\rangle_B - \frac{\lambda_R}{2}\phi\phi'\langle\psi^2\rangle_{ad}
$$
$$
= \frac{\lambda_R}{2}\phi\phi'[\langle\psi^2\rangle_B - \langle\psi^2\rangle_R - \langle\psi^2\rangle_{ad}] = 0, \quad \text{(B.12)}
$$

where the last identity follows from the definition of $\langle\psi^2\rangle_R$.

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